2020

MATHEMATICS

(*Metric Space and Complex Analysis - IV*)

[Honours]

Paper – XIII

Full Marks : 80

Time : 3 hours

Answer **all** questions

The figures in the right-hand margin indicate marks

GROUP – A

1. Answer any *ten* questions of the following : 2×10

(*a*) Let X is a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

For all $x, y \in X$. Show that d is a metric on $\mathbb{R}$.

(Turn Over)

(*b*) Express $f(z) = z + \frac{1}{z}$ in form of $u(r, \theta)$ and $v(r, \theta)$.

(*c*) Define Dirichlet function.

(*d*) Find the derivative of $f(z) = z^2 + 3z + 1$

(*e*) Show that l^p is separable.

(*f*) Show that $\lim_{z \to 0} \frac{1}{z^2} = \infty$

(*g*) Show that Euclidean space is a complete metric space.

(*h*) Sketch the region given by $|\operatorname{Im} Z| > 1$.

(*i*) Let (X, d) is a metric space and $A, B \subset X$ are disjoint set. Show that if A and B are both open, then they are separated.

(*j*) Find the square root of $-7 - 24i$.

(*k*) Define Lipschitz constant.

(*l*) If Z_1, Z_2, Z_3 are the vertices of an equilateral triangle then show that

$$Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$$

GROUP – B

Answer either (*a*, *b*) or (*c*, *d*) of the following questions : 20 × 3

2. (*a*) Prove that every nonempty open set in the usual metric space is the union of countable disjoint class of open interval. 10

(*b*) Let X be the set of all real valued continuous function on $J = [0, 1]$ and let

$$d(x, y) = \int_0^1 |x(t) - y(t)|\, dt.$$

Show that (X, d) is not complete. 10

Or

(*c*) **State and prove Hausdorff theorem.** 10

(*d*) **Let (X, d) is a complete metric space. If (G_n)**

is a sequence of dense open sets in X then prove that $G = \bigcup_{n=1}^{\infty} G_n$ is dense in X. 10

3. (*a*) Let (X, d) and (Y, s) are two metric spaces and $f : X \to Y$ is a bijective function. Then the following statements are equivalent.

(*i*) F is a homeomorphism.

(*ii*) The set $G \subset X$ is open if and only if its image $F(G) \subset Y$ is open.

(*iii*) The set $F \subset X$ is closed if and only if its image $f(F) \subset Y$ is closed. 10

(*b*) Let (X, d) and (Y, s) are two metric space and $f : X \to Y$ and $g : X \to Y$ are two continuous function then $\{x \in X : f(x) = g(x)\}$ is closed. 10

Or

(*c*) **State and prove Banach fixed point theorem.** 10

(*d*) Let (X, d) and (Y, s) are two metric space and $f: X \to Y$ is a function. Then f is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$ for every subset A of X. 10

4. (*a*) Show that the given function satisfies Cauchy Riemann equation at origin but is not differentiable there.

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$ 10

(*b*) Prove that every differentiable function is continuous but the converse is not true. 10

Or

(*c*) Derive the equation of circle passing through 3 points. Find the equation of the circle passing through $(1, i, 1 + i)$. 10

(*d*) Derive polar form of Cauchy Riemann equation and show that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$$

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}$$ 10