2021

Full Marks : 80

Time : 3 hours

The figures in the right-hand margin indicate marks

Answer **all** questions

1. Answer *all* questions : 12×1

(a) Any group of order 3 is a ——— group.

(b) If $\mathbb{Z}_3$ the set of residue class of modulo 3, what is the additive inverse of 2 ?

(c) If $G = \{1, w, w^2\}$, w is cube root of unity, what is generator of G ?

(d) Is $\mathbb{R}$ the set of real number is a group under multiplication?

(e) If it is a subgroup of the group G, $o(H) = 3$, $o(G) = 12$, what is index of H in G ?

(f) If H is a subset of the group G, then it is a subgroup of G if ———.

(g) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2^x$. Is f is a homomorphism?

(h) In the group of set of complex numbers, multiplicative inverse of $1 + 2i$ is ———.

(i) If $G = \{a, a^2, a^3, \cdots, a^8 = e\}$, then subgroup of G generated by a^2 is ———.

(j) If group G is isomorphic to the group G' and order of G is n, then order of G' is ———.

(k) If G is a group, $a \in G$, normaliser (a) = centraliser of $G = G$ of G is ——— group.

(l) The set of permutations of three symbols is a group of order ———.

2. Define *any eight* of the following each with an example : 8×2

(a) Abelian group

(b) Cyclic group

(c) Normaliser of g, $g \in G$, G is a group

(d) Centralizer of the group G

(e) Normal subgroup of the group G

(f) Homomorphism

(g) Isomorphism

(h) Endomorphism

(i) Cartesian product of two subgroups of the group G

(j) Right coset of the group G.

3. Answer *any eight* questions : 8×3

(a) In a group $(G, +)$, prove that—

$$x + y = x + z \Rightarrow y = z,\ x, y, z \in G$$

(b) In a group $(G, \times)$, prove that—

$$(x^{-1})^{-1} = x,\ x \in G$$

(c) If $(ab)^2 = a^2 b^2$, $\forall a, b \in G$, then prove that G is an Abelian group.

(d) If $f : G \to \overline{G}$ is an isomorphism, then prove that $\vec{f} : \overline{G} \to G$ is an isomorphism.

(e) Prove that every cyclic group is an Abelian group.

(f) If

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

then find $(f \circ g)^{-1}$.

(g) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 3 & 5 & 7 & 6 \end{pmatrix}$

Express f as the product of disjoint cycles. Is it an even or odd permutation? Justify your answer.

(h) Prove that the subgroup N of the group G is a normal subgroup if $N_a \cdot N_b = N_{ab}$, $a, b \in G$.

(i) If $f : G \to G$ is a homomorphism, then prove that kernel of f is a normal subgroup of G.

4. Answer *all* questions : 4×7

(a) Define Abelian group. Prove that the set of real numbers form an Abelian group under + operation.

Or

Prove that the set of all permutations of three symbols 1, 2, 3 forms a group under composition mapping. Is it an Abelian group? Justify your answer.

(b) If H is a subgroup of the group G, then prove that $o(H)$ divides $o(G)$.

Or

If f is a homomorphism from group G to $\overline{G}$, then prove that $G/\text{kernel } f$ is isomorphic to $f(G)$.

(c) Show that

$$GL(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in \mathbb{R},\ ad - bc \neq 0 \right\}$$

is a group under matrix multiplication. Is it an Abelian group? Justify your answer.

Or

If G is a finite Abelian group, p is a prime number such that p divides $o(G)$, then prove that there exist $a \neq e \in G$ such that $a^p = e$.

(d) If H is a subset of the group G, then prove that H is a subgroup of G if and only if $a, b \in H \Rightarrow ab^{-1} \in H$.

Or

If H is a normal subgroup of the group G and K is a normal subgroup of G containing H, then prove that G/K is isomorphic to $\dfrac{(G/H)}{(K/H)}$.

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